表面に働く靜的力に依る有層半無限彈性體の歪に就て (第一報)

副　田　勝　利*

半無限彈性體の歪に關する研究は J. Boussinesq 以後多くの人に依つてなされて來て居る。靜的力が等方半無限彈性體の表面に作用する一般の場合は 1935 年本多，三浦兩氏[1]のものがあり，有層半無限彈性體の表面荷重に依る歪の研究は工學的見地から數値積分を用ひて計算をした松村孫治氏[2]の研究がある．その後層が極めて薄いとして計算をなしたものに西村源六郎氏[3]及澤田龍吉氏[4]の研究と，又下層が剛體とした場合の同じ澤田龍吉氏の研究がある．

著者は有層半無限彈性體の表面に靜的力が働く一般の場合に就て計算をなした．

1. 圓壔座標に依る彈性體の平衡方程式の解

圓壔座標に於て r, θ, z 方向の變位を u_r, u_θ, u_z とすると，平衡方程式は

$$\left.\begin{aligned}
&\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}+\frac{\partial^2 u_r}{\partial z^2}-\frac{1}{r^2}\left(u_r-\frac{\partial^2 u_r}{\partial\theta^2}\right)-\frac{2}{r^2}\cdot\frac{\partial u_\theta}{\partial\theta}=-\frac{\lambda+\mu}{\mu}\cdot\frac{\partial\Delta}{\partial r},\\
&\frac{\partial^2 u_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial u_\theta}{\partial r}+\frac{\partial^2 u_\theta}{\partial z^2}-\frac{1}{r^2}\left(u_\theta-\frac{\partial^2 u_\theta}{\partial\theta^2}\right)+\frac{2}{r^2}\cdot\frac{\partial u_r}{\partial\theta}=-\frac{\lambda+\mu}{\mu}\cdot\frac{1}{r}\frac{\partial\Delta}{\partial\theta},\\
&\frac{\partial^2 u_z}{\partial r^2}+\frac{1}{r}\frac{\partial u_z}{\partial r}+\frac{\partial^2 u_z}{\partial z^2}+\frac{1}{r^2}\cdot\frac{\partial^2 u_z}{\partial\theta^2}=-\frac{\lambda+\mu}{\mu}\cdot\frac{\partial\Delta}{\partial z}.
\end{aligned}\right\}\cdots\cdots\cdots\cdots(1)$$

こゝに z は下向を正に取る，λ, μ はラーメの常數で，Δ は次の式を滿足する體積伸張である．

$$\frac{\partial^2\Delta}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial\Delta}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Delta}{\partial\theta^2}+\frac{\partial^2\Delta}{\partial z^2}=0 \quad\cdots\cdots\cdots\cdots(2)$$

この方程式の解は寺澤博士[5]が解いて居られる如く，次の樣になる．

$\Delta=(P_{1m}e^{-kz}+P'_{1m}e^{kz})J_m(kr)\cos m\theta$ の場合

$$u_r=\left[\left\{\left(\frac{\lambda+\mu}{2}P_{1m}z-Q_{1m}\right)e^{-zk}+\left(-\frac{\lambda+\mu}{2\mu}P'_{1m}z-Q'_{1m}\right)e^{kz}\right\}J'_m(kr)+(R_{1m}e^{-kz}+R'_{1m}e^{kz})\frac{m}{kr}J_m(kr)\right]\cos m\theta,$$

---

* 中央氣象臺

1) H. Honda and T. Miura; Geo. Mag. Vol. IX 1935.

2) M. Matumura; Jour. Civil. Eng. Soc. Tokyo **17** (1931) No. 11.

3) G. Nisimura; 震研彙報，第 10 號，第 1 册，1932.

4) 澤田龍吉; 氣象集誌，第 2 輯，第 15 卷第 9 號，第 16 卷第 1 號.

5) K. Terazawa; Journ. Coll. Sci. Imp. Uni. Tokyo Vol. 37 (1916)

$$u_\theta=\left[\left\{\left(-\frac{\lambda+\mu}{2\mu}P_{1m}z+Q_{1m}\right)e^{-kz}+\left(\frac{\lambda+\mu}{2\mu}P'_{1m}z+Q'_{1m}\right)e^{kz}\right\}\frac{m}{kr}J_m(kr)\right.$$
$$\left.-(R_{1m}e^{-kz}+R'_{1m}e^{kz})J'_m(kr)\right]\sin m\theta;$$
$$u_z=-\left\{\left(\frac{\lambda+\mu}{2\mu}P_{1m}z-S_{1m}\right)e^{-kz}+\left(\frac{\lambda+\mu}{2\mu}P'_{1m}z-S'_{1m}\right)e^{kz}\right\}J_m(kr)\cos m\theta$$
$$\frac{\lambda+3\mu}{2\mu}P_{1m}=k(Q_{1m}-S_{1m}),\qquad \frac{\lambda+3\mu}{2\mu}P'_{1m}=k(Q'_{1m}+S'_{1m})\qquad\cdots\cdots(3)$$

$\Delta=(\overline{P}_{1m}e^{-kz}+\overline{P'}_{1m}e^{kz})J_m(kr)\sin m\theta$ の場合

$$\bar{u}_r=\left[\left\{\left(\frac{\lambda+\mu}{2\mu}\overline{P}_{1m}z-\overline{Q}_{1m}\right)e^{-kz}+\left(-\frac{\lambda+\mu}{2\mu}\overline{P}'_{1m}z-\overline{Q}'_{1m}\right)e^{kz}\right\}J'_m(kr)\right.$$
$$\left.+\{\overline{R}_{1m}e^{-kz}+\overline{R}'_{1m}e^{kz}\}\frac{m}{kr}J_m(kr)\right]\sin m\theta.$$
$$\bar{u}_\theta=\left[\left\{\left(\frac{\lambda+\mu}{2\mu}\overline{P}_{1m}z-\overline{Q}_{1m}\right)e^{-kz}+\left(-\frac{\lambda+\mu}{2\mu}\overline{P}'_{1m}z-\overline{Q}'_{1m}\right)e^{kz}\right\}\frac{m}{kr}J_m(kr)\right.$$
$$\left.+\{\overline{R}_{1m}e^{-kz}+\overline{R}'_{1m}e^{kz}\}J'_m(kr)\right]\cos m\theta.$$
$$\bar{u}_z=-\left\{\left(\frac{\lambda+\mu}{2\mu}\overline{P}_{1m}z-\overline{S}_{1m}\right)e^{-kz}+\left(\frac{\lambda+\mu}{2\mu}\overline{P}'_{1m}z-\overline{S}'_{1m}\right)e^{kz}\right\}J_m(kr)\sin m\theta$$
$$\frac{\lambda+3\mu}{2\mu}\overline{P}_{1m}=k(\overline{Q}_{1m}-\overline{S}_{1m}),\qquad \frac{\lambda+3\mu}{2\mu}\overline{P}'_{1m}=k(\overline{Q}'_{1m}+\overline{S}_{1m}).\qquad\cdots\cdots(4)$$

$P_{1m},\ Q_{1m},\ S_{1m},\ R_{1m}\cdots$ 等は積分常數である.

次に歪力成分は次式の公式に依つて求められる.

$$\widehat{zz}=\lambda\Delta+2\mu\frac{\partial u_z}{\partial z},$$
$$\widehat{zr}=\mu\left\{\frac{\partial u_z}{\partial z}+\frac{\partial u_z}{\partial r}\right\}\qquad\cdots\cdots(5)$$
$$\widehat{z\theta}=\mu\left\{\frac{1}{r}\frac{\partial u_z}{\partial\theta}+\frac{\partial u_\theta}{\partial z}\right\}$$

即ち

$$\widehat{zz}=[\{(\lambda+\mu)P_{1m}kz-2\mu kS_{1m}-\mu P_{1m}\}e^{-kz}+\{-(\lambda+\mu)P'_{1m}kz$$
$$+2\mu kS'_{1m}-\mu P'_{1m}\}e^{kz}]J_m(kr)\cos m\theta$$
$$\widehat{zr}=-\{[\{(\lambda+\mu)P_{1m}kz-\frac{\lambda+\mu}{2}P_{1m}-\mu k(S_{1m}+Q_{1m})\}e^{-kz}$$
$$+\{(\lambda+\mu)P'_{1m}kz+\frac{\lambda+\mu}{2}P'_{1m}+\mu k(Q'_{1m}-S'_{1m})\}e^{kz}]J'_m(kr)$$
$$+\frac{\mu m}{r}(R_{1m}e^{-kz}-R'_{1m}e^{kz})J_m(kr)\}\cos m\theta.\qquad\cdots\cdots(6)$$

$$\left.\begin{aligned}
\widehat{z\theta}=&[[\{(\lambda+\mu)P_{1m}kz-\frac{\lambda+\mu}{2}P_{1m}-\mu k(S_{1m}+Q_{1m})\}e^{-kz}\\
&+\{(\lambda+\mu)P'_{1m}kz+\frac{\lambda+\mu}{2}P'_{1m}+\mu k(Q'_{1m}-S'_{1m})\}e^{kz}]\frac{m}{kr}J_m(kr)\\
&+\mu k(R_{1m}e^{-kz}-R'_{1m}e^{kz})J'_m(kr)\sin m\theta
\end{aligned}\right.$$

$$\left.\begin{aligned}
\overline{\widehat{zz}}=&[\{(\lambda+\mu)\overline{P}_{1m}kz-2\,\mu k\overline{S}_{1m}-\mu\overline{P}_{1m}\}e^{-kz}+\{-(\lambda+\mu)\overline{P'}_{1m}kz\\
&+2\,\mu k\overline{S'}_{1m}-\mu\overline{P'}_{1m}\}e^{kz}]J_m(kr)\sin m\theta.\\
\overline{\widehat{zr}}=&-\{[\{(\lambda+\mu)\overline{P}_{1m}kz-\frac{\lambda+\mu}{2}\overline{P}_{1m}-k\mu(\overline{Q}_{1m}+\overline{S}_{1m})\}e^{-kz}\\
&+\{(\lambda+\mu)\overline{P'}_{1m}kz+\frac{\lambda+\mu}{2}\overline{P'}_{1m}+\mu k(\overline{Q'}_{1m}-\overline{S'}_{1m})\}e^{kz}]J'_m(kr)\\
&+\{\overline{R}_{1m}e^{-kz}-\overline{R'}_{1m}e^{kz}\}\frac{\mu m}{r}J_m(kr)\}\sin m\theta.\\
\overline{\widehat{z\theta}}=&\{[\{-(\lambda+\mu)\overline{P}_{1m}z+\frac{\lambda+\mu}{2}\overline{P}_{1m}+\mu k(\overline{Q}_{1m}+\overline{S}_{1m})\}e^{-kz}\\
&+\{-(\lambda+\mu)\overline{P'}_{1m}z-\frac{\lambda+\mu}{2}\overline{P'}_{1m}-\mu k(\overline{Q'}_{1m}-\overline{S'}_{1m})\}e^{kz}]\frac{m}{kr}J_m(kr)\\
&-\mu k\{\overline{R}_{1m}e^{-kz}-\overline{R'}_{1m}e^{kz}\}J'_m(kr)\}\cos m\theta
\end{aligned}\right\}\cdots\cdots(7)$$

2.　有層半無限彈性體の場合

表面層の厚さを f, そのラーメ常數を λ_1, μ_1 下層のラーメ常數を λ_2, μ_2 とし，積分常數の添數を 1 及 2 でもつて表はす事にする．下層に於て $z=\infty$ では變位$=0$ にならなければならぬから，P'_{2m}, S'_{2m}, Q'_{2m}' R'_{2m}, 及び $\overline{P'}_{2m}$, $\overline{S'}_{2m}$, $\overline{Q'}_{2m}$, $\overline{R'}_{2m}$ は零である．

即ち表面層に於ける變位及び歪力成分は

$$\left.\begin{aligned}
u_r=&\left[\left\{\left(\frac{\lambda_1+\mu_1}{2\mu_1}P_{1m}z-Q_{1m}\right)e^{-kz}+\left(-\frac{\lambda_1+\mu_1}{2\mu_1}P'_{1m}z-Q'_{1m}\right)e^{kz}\right\}J'_m(kr)\right.\\
&\left.+(R_{1m}e^{-kz}+R'_{1m}e^{kz})\frac{m}{kr}J_m(kr)\right]\cos m\theta,\\
u_\theta=&\left[\left\{\left(-\frac{\lambda_1+\mu_1}{2\mu_1}P_{1m}z+Q_{1m}\right)e^{-kz}+\left(\frac{\lambda_1+\mu_1}{2\mu_1}P'_{1m}z+Q'_{1m}\right)c^{kz}\right\}\frac{m}{kr}J_m(kr)\right.\\
&\left.-(R_{1m}e^{-kz}+R'_{1m}e^{kz})J'_m(Kr)\right]\sin m\theta,\\
u_z=&-\left\{\left(\frac{\lambda_1+\mu_1}{2\mu_1}P_{1m}z-S_{1m}\right)e^{-kz}+\left(\frac{\lambda_1+\mu_1}{2\mu_1}P'_{1m}z-S'_{1m}\right)e^{kz}\right\}J_m(kr)\cos m\theta.
\end{aligned}\right\}\cdot\cdot(8)$$

$$\left.\begin{aligned}
\bar{u}_r=&\left[\left\{\left(\frac{\lambda_1+\mu_1}{2\mu_1}\overline{P}_{1m}z-\overline{Q}_{1m}\right)e^{-kz}+\left(-\frac{\lambda_1+\mu_1}{2\mu_1}\overline{P'}_{1m}z-\overline{Q'}_{1m}\right)e^{kz}\right\}J'_m(kr)\right.\\
&\left.+\{\overline{R}_{1m}e^{-kz}+\overline{R'}_{1m}e^{kz}\}\frac{m}{kr}J_m(kr)\right]\sin m\theta.
\end{aligned}\right.$$

$$\left.\begin{aligned}\bar{u}_\theta=&\left[\left\{\left(\frac{\lambda_1+\mu_1}{2\mu_1}\bar{P}_{1m}z-\bar{Q}_{1m}\right)e^{-kz}+\left(-\frac{\lambda_1+\mu_1}{2\mu_1}\bar{P}'_{1m}z-\bar{Q}'_{1m}\right)e^{kz}\right\}\frac{m}{kr}J_m(kr)\right.\\&\left.+\{\bar{R}_{1m}e^{-kz}+\bar{R}'_{1m}e^{kz}\}J'_m(kr)\right]\cos m\theta,\\\bar{u}_z=&-\left\{\left(\frac{\lambda_1+\mu_1}{2\mu_1}\bar{P}_{1m}z-\bar{S}_{1m}\right)e^{-kz}+\left(\frac{\lambda_1+\mu_1}{2\mu_1}\bar{P}'_{1m}z-\bar{S}'_{1m}\right)e^{kz}\right\}J_m(kr)\sin m\theta.\end{aligned}\right\}\cdot\cdot(9)$$

$$\left.\begin{aligned}\widehat{zz}=&[\{(\lambda_1+\mu_1)P_{1m}kz-2\mu_1kS_{1m}-\mu_1P_{1m}\}e^{-kz}\\&+\{-(\lambda_1+\mu_1)P'_{1m}kz+2\mu_1kS'_{1m}-\mu_1P'_{1m}\}e^{kz}]J_m(kr)\cos m\theta,\\\widehat{zr}=&-\{[\{(\lambda_1+\mu_1)P_{1m}kz-\frac{\lambda_1+\mu_1}{2}P_{1m}-\mu_1k(S_{1m}+Q_{1m})\}e^{-kz}\\&+\{(\lambda_1+\mu_1)P'_{1m}kz+\frac{\lambda_1+\mu_1}{2}P'_{1m}+\mu_1k(Q'_{1m}-S'_{1m})\}e^{kz}]J'_m(kr)\\&+\frac{\mu_1m}{r}(R_{1m}e^{-kz}-R'_{1m}e^{kz})J_m(kr)\}\cos m\theta,\\\widehat{z\theta}=&\{[\{(\lambda_1+\mu_1)P_{1m}kz-\frac{\lambda_1+\mu_1}{2}P_{1m}-\mu_1k(S_{1m}+Q_{1m})\}e^{-kz}\\&+\{(\lambda_1+\mu_1)P'_{1m}kz+\frac{\lambda_1+\mu_1}{2}P'_{1m}+\mu_1k(Q'_{1m}-S'_{1m})\}e^{kz}]\frac{m}{kr}J_m(kr)\\&+\mu_1k(R_{1m}e^{-kz}-R'_{1m}e^{kz})J'_{1m}(kr)\}\sin m\theta.\end{aligned}\right\}\cdot\cdot(10)$$

$$\left.\begin{aligned}\overline{\widehat{zz}}=&[\{(\lambda_1+\mu_1)\bar{P}_{1m}kz-2\mu_1K\bar{S}_{1m}-\mu_1\bar{P}_{1m}\}e^{-kz}\\&+\{-(\lambda_1+\mu_1)\bar{P}'_{1m}kz+2\mu_1k\bar{S}'_{1m}-\mu_1\bar{P}'_{1m}\}e^{kz}]J_m(kr)\sin m\theta,\\\overline{\widehat{zr}}=&-\{[\{(\lambda_1+\mu_1)\bar{P}_{1m}kz-\frac{\lambda_1+\mu_1}{2}P_{1m}k-\mu_1(Q_{1m}+S_{1m})\}e^{-kz}\\&+\{\lambda_1+\mu_1)\bar{P}'_{1m}kz+\frac{\lambda_1+\mu_1}{2}\bar{P}'_{1m}+\mu_1k(\bar{Q}'_{1m}-\bar{S}'_{1m})\}e^{kz}]J'_m(kr)\\&+\{\bar{R}_{1m}e^{-kz}-\bar{R}'_{1m}e^{kz}\}\frac{\mu_1m}{r}J_m(kr)\}\sin m\theta,\\\overline{\widehat{z\theta}}=&\{[\{-(\lambda_1+\mu_1)\bar{P}_{1m}kz+\frac{\lambda_1+\mu_1}{2}\bar{P}_{1m}+\mu_1k(\bar{Q}_{1m}+\bar{S}_{1m})\}e^{-kz}\\&+\{-(\lambda_1+\mu_1)\bar{P}'_{1m}z-\frac{\lambda_1+\mu_1}{2}\bar{P}'_{1m}-\mu_1k(\bar{Q}'_{1m}-\bar{S}'_{1m})\}e^{kz}]\frac{m}{kr}J_m(kr)\\&-\mu_1k\{\bar{R}_{1m}e^{-kz}-\bar{R}'_{1m}e^{kz}\}J'_m(kr)\}\cos m\theta.\end{aligned}\right\}\cdot\cdot(11)$$

下層の變位及歪力成分は

$$u_r=\left[\left\{\left(\frac{\lambda_2+\mu_2}{2\mu_2}P_{2m}z-Q_{2m}\right)e^{-kz}J'_m(kr)+R_{2m}e^{-kz}\frac{m}{kr}J_m(kr)\right]\cos m\theta,$$

$$\left.\begin{aligned}u_\theta&=\left[\left\{\left(-\frac{\lambda_2+\mu_2}{2\mu_2}P_{2m}z+Q_{2m}\right)e^{-kz}\frac{m}{kr}J_m(kr)-R_{2m}e^{-kz}J'_m(kr)\right]\sin m\theta,\\u_z&=-\left(\frac{\lambda_2+\mu_2}{2\mu_2}P_{2m}z-S_{2m}\right)e^{-kz}J_m(kr)\cos m\theta.\end{aligned}\right\}\cdots\cdots(12)$$

$$\left.\begin{aligned}\bar{u}_r&=\left[\left\{\left(\frac{\lambda_2+\mu_2}{2\mu_2}\overline{P}_{2m}z-\overline{Q}_{2m}\right)e^{-kz}J'_m(kr)+\overline{R}_{1m}e^{-kz}\frac{m}{kr}J_m(kr)\right]\sin m\theta,\\\bar{u}_\theta&=\left[\left(\frac{\lambda_2+\mu_2}{2\mu_2}\overline{P}_{2m}z-\overline{Q}_{2m}\right)e^{-kz}\frac{m}{kr}J_m(kr)+\overline{R}_{1m}e^{-kz}J'_m(kr)\right]\cos m\theta,\\\bar{u}_z&=-\left(\frac{\lambda_2+\mu_2}{2\mu_2}\overline{P}_{2m}z-\overline{S}_{1m}\right)e^{-kz}J_m(kr)\sin m\theta.\end{aligned}\right\}\cdots\cdots(13)$$

$$\left.\begin{aligned}\widehat{zz}&=\{(\lambda_2+\mu_2)P_{2m}kz-2\mu_2kS_{2m}-\mu_2P_{2m}\}e^{-kz}J_m(kr)\cos m\theta,\\\widehat{zr}&=[\{(\lambda_2+\mu_2)P_{2m}kz-\frac{\lambda_2+\mu_2}{2}P_{2m}-\mu_2k(S_{2m}+Q_{2m})\}e^{-kz}J'_m(kr)\\&\quad+\frac{\mu_2m}{r}R_{2m}e^{-kz}J_m(kr)]\cos m\theta,\\\widehat{z\theta}&=[\{(\lambda_2+\mu_2)P_{2m}kz-\frac{\lambda_2+\mu_2}{2}P_{2m}-\mu_2k(S_{2m}+Q_{2m})\}e^{-kz}\frac{m}{kr}J_m(kr)\\&\quad+\mu_2k\overline{R}_{2m}e^{-kz}J'_m(kr)]\sin m\theta,\end{aligned}\right\}\cdots\cdots(14)$$

$$\left.\begin{aligned}\overline{\widehat{zz}}&=\{(\lambda_2+\mu_2)\overline{P}_{2m}kz-2\mu_2k\overline{S}_{2m}-\overline{\mu}_2P_{2m}\}e^{-kz}J_m(kr)\sin m\theta,\\\overline{\widehat{zr}}&=-[\{(\lambda_2+\mu_2)\overline{P}_{2m}kz-\frac{\lambda_2+\mu_2}{2}\overline{P}_{2m}-k\mu_2(\overline{Q}_{2m}+\overline{S}_{2m})\}e^{-kz}J'_m(kr)\\&\quad+\overline{R}_{1m}e^{-kz}\frac{\mu_2m}{r}J_m(kr)]\sin m\theta,\\\overline{\widehat{z\theta}}&=[\{-(\lambda_2+\mu_2)\overline{P}_{2m}kz+\frac{\lambda_2+\mu_2}{2}\overline{P}_{2m}+\mu_2k(\overline{Q}_{2m}+\overline{S}_{2m})\}e^{-kz}\frac{m}{kr}J_m(kr)\\&\quad-\mu_2k\overline{R}_{2m}e^{-kz}J'_m(kr)]\cos m\theta.\end{aligned}\right\}\cdots\cdots(15)$$

今表面に働く靜的力を次の様に表す．但し $\Pi_m(r)$, $\Phi_m(r)$, $Z_m(r)\cdots\cdots$等は r のみの函數とし F_r, F_θ, F_z は夫々 r, θ, z 方向の歪力成分とする

$$\left.\begin{aligned}F_r(r,\theta)&=\Pi_m(r)\cos m\theta+\overline{\Pi}_m(r)\sin m\theta\\F_\theta(r,\theta)&=\Phi_m(r)\cos m\theta+\overline{\Phi}_m(r)\sin m\theta\\F_z(r,\theta)&=Z_m(r)\cos m\theta+\overline{Z}_m(r)\sin m\theta\end{aligned}\right\}\cdots\cdots(16)$$

$$\widehat{zr}=-F_r,\qquad\widehat{z\theta}=-F_\theta,\qquad\widehat{zz}=-F_z$$

と置くと，此等は次の如く展開される．

$$\frac{\widehat{zr}}{\mu_1}=-\frac{1}{\mu_1}\int_0^\infty\{S_m(k)J_{m+1}(kr)+T_m(k)J_{m-1}(kr)\}kdk\cos m\theta$$

$$\left.\begin{aligned}\frac{\widehat{z\theta}}{\mu_1}&=-\frac{1}{\mu_1}\int_0^\infty\{S_m(k)J_{m+1}(kr)-T_m(k)J_{m-1}(kr)\}kdk\sin m\theta\\ \frac{\widehat{zz}}{\mu_1}&=-\frac{1}{\mu_1}\int_0^\infty W_m(k)J_m(kr)kdk\cos m\theta\end{aligned}\right\}\cdots\cdots(17)$$

$$\left.\begin{aligned}\frac{\overline{\widehat{zr}}}{\mu_1}&=-\frac{1}{\mu_1}\int_0^\infty\{\overline{S}_m(k)J_{m+1}(kr)+\overline{T}_m(k)J_{m-1}(kr)\}kdk\sin m\theta\\ \frac{\overline{\widehat{z\theta}}}{\mu_1}&=\frac{1}{\mu_1}\int_0^\infty\{\overline{S}_m(k)J_{m+1}(kr)-T_m(k)J_{m-1}(kr)\}kdk\cos m\theta\\ \frac{\overline{\widehat{zz}}}{\mu_1}&=-\frac{1}{\mu_1}\int_0^\infty\overline{W}_m(k)J_m(kr)kdk\sin m\theta\end{aligned}\right\}\cdots\cdots(18)$$

こゝに

$$S_m(k)=\frac{1}{2}\int_0^\infty\{\Pi_m(\varpi)+\overline{\Phi}_m(\varpi)\}J_{m+1}(k\varpi)\varpi d\varpi,$$

$$T_m(k)=\frac{1}{2}\int_0^\infty\{\Pi_m(\varpi)-\overline{\Phi}_m(\varpi)\}J_{m-1}(k\varpi)\varpi d\varpi,$$

$$W_m(k)=\int_0^\infty Z_m(\varpi)J_m(k\varpi)\varpi d\varpi,$$

$$\overline{S}_m(k)=\frac{1}{2}\int_0^\infty\{\overline{\Pi}_m(\varpi)-\Phi_m(\varpi)\}J_{m+1}(k\varpi)\varpi d\varpi,$$

$$\overline{T}_m(k)=\frac{1}{2}\int_0^\infty\{\overline{\Pi}_m(\varpi)+\Phi_m(\varpi)\}J_{m-1}(k\varpi)\varpi d\varpi,$$

$$\overline{W}_m(k)=\int_0^\infty\overline{Z}_m(\varpi)J_m(k\varpi)\varpi d\varpi.$$

3. 積分常數の決定

こゝで表面に働く力並に層の境界に於て變位及歪力は連續であると云ふ條件を滿足する樣に常數を決定しなければならない. 先ず (8), (10), (12), (14) の各式から

$$\left.\begin{aligned}&\{(\lambda_1+\mu_1)P_{1m}kf-2\mu_1kS_{1m}-\mu_1P_{1m}\}e^{-kf}\\ &\quad+\{-(\lambda_1+\mu_1)P'_{1m}kf+2\mu_1kS'_{1m}-\mu_1P'_{1m}\}e^{kf}\\ &\quad=\{(\lambda_2+\mu_2)P_{2m}kf-2\mu_2kS_{2m}-\mu_2P_{2m}\}e^{-kf}\\ &\{(\lambda_1+\mu_1)P_{1m}kf-\frac{\lambda_1+\mu_1}{2}P_{1m}-\mu_1k(S_{1m}+Q_{1m})\}e^{-kf}\\ &\quad+\{(\lambda_1+\mu_1)P'_{1m}kf+\frac{\lambda_1+\mu_1}{2}P'_{1m}+\mu_1k(Q'_{1m}-S'_{1m})\}e^{kf}\\ &\quad=\{(\lambda_2+\mu_2)P_{2m}kf-\frac{\lambda_2+\mu_2}{2}P_{2m}-\mu_2k(S_{2m}+Q_{2m})\}e^{-kf}\\ &\quad\mu_1(R_{1m}e^{-kf}-R'_{1m}e^{kf})=\mu_2R_{2m}e^{-kf}\end{aligned}\right\}\cdots\cdots(19)$$

$$\left.\begin{aligned}&\left(\frac{\lambda_1+\mu_1}{2\mu_1}P_{1m}f-Q_{1m}\right)e^{-kf}+\left(-\frac{\lambda_1+\mu_1}{2\mu_1}P'_{1m}f-Q'_{1m}\right)e^{kf}\\&\qquad=\left(\frac{\lambda_2+\mu_2}{2\mu_2}P_{2m}f-Q_{2m}\right)e^{-kf}\\&\qquad R_{1m}e^{-kf}+R'_{1m}e^{kf}=R_{2m}e^{-kf}\\&\left(\frac{\lambda_1+\mu_1}{2\mu_1}P_{1m}f-S_{1m}\right)e^{-kf}+\left(\frac{\lambda_1+\mu_1}{2\mu_1}P''_{1m}f-S'_{1m}\right)e^{kf}\\&\qquad=\left(\frac{\lambda_2+\mu_2}{2\mu_2}P_{2m}f-S_{2m}\right)e^{-kf}\end{aligned}\right\}\cdots\cdots(20)$$

$$\left.\begin{aligned}&-\frac{\lambda_1+\mu_1}{2}P_{1m}-\mu_1k(S_{1m}+Q_{1m})+\frac{\lambda_1+\mu_1}{2}P'_{1m}\\&\qquad+\mu_1k(Q'_{1m}-S'_{1m})=-k\{S_m(k)-T_m(k)\}\\&\qquad\mu_1(R_{1m}-R'_{1m})=S_m(k)+T_m(k)\\&-2\mu_1kS_{1m}-\mu_1P_{1m}+2\mu_1kS_{1m}-\mu_2P'_{1m}=-kW_m(k)\end{aligned}\right\}\cdots\cdots(21)$$

此等から常數を決定するのは非常に面倒であるから，今後は $\lambda_1=\mu_1$, $\lambda_2=\mu_2$ なる場合について計算する．且 $\mu_2/\mu_1=n$ と置くと

$$\begin{aligned}P_{1m}=\Big[&4\frac{k}{\mu_1}\{S_m(k)-T_m(k)\}\{-(n+2)(1+2n)e^{2kf}+(1-n)(n+2)\\&+2(1-n)(2+n)kf\}+4\frac{k}{\mu_1}W_m(k)\{(n+2)(1+2n)e^{2kf}\\&-(1-n)(n+2)+2(1-n)(n+2)kf\}\Big]\times\Delta^{-1}\end{aligned}$$

$$\begin{aligned}S_{1m}=\Big[&-\frac{1}{\mu_1}\{S_m(k)-T_m(k)\}\{-2(n+2)(2n+1)e^{2kf}+4(1-n)^2+4(1-n)(n+2)kf\\&-8(1-n)(n+2)k^2f^2\}+\frac{1}{\mu_1}W_m(k)\{-6(n+2)(2n+1)e^{2kf}\\&-12(1-n)(n+3)-12(n+2)(1-n)kf+8(1-n)(n+2)k^2f^2\}\Big]\times\Delta^{-1}\end{aligned}$$

$$\begin{aligned}Q_{1m}=\Big[&\frac{1}{\mu_1}\{S_m(k)-T_m(k)\}\{-6(n+2)(2n+1)e^{2kf}-12(1-n^2)+12(1-n)(2+n)kf\\&+8(1-n)(n+2)k^2f^2\}+\frac{1}{\mu_1}W_m(k)\{2(n+2)(2n+1)e^{2kf}\\&-4(1-n)^2+4(1-n)(n+2)kf+8(1-n)(2+n)k^2f^2\}\Big]\times\Delta^{-1}\end{aligned}$$

$$\begin{aligned}P'_{1m}=\Big[&\frac{k}{\mu_1}\{S_m(k)-T_m(k)\}\{8(1-n)^2e^{-2kf}-4(1-n)(n+2)+8(n+2)(1-n)kf\}\\&-\frac{k}{\mu_1}W_m(k)\{-8(1-n)^2e^{-2kf}+4(1-n)(n+2)+8(n+2)(1-n)kf\}\Big]\times\Delta^{-1}\end{aligned}$$

$$S'_{1m}=[\frac{1}{\mu_1}\{S_m(k)-T_m(k)\}\{4(1-n)^2e^{-2kf}+4(1-n)^2+4(1-n)(n+2)kf$$

$$+8(1-n)(2+n)k^2f^2\}+\frac{1}{\mu_1}W_m(k)\{12(1-n)^2e^{-2kf}+12(n^2-1)$$

$$-12(1-n)(n+2)kf-8(1-n)(2+n)k^2f^2\}]\times\Delta^{-1}$$

$$Q'_{1m}=[\frac{1}{\mu_1}\{S_m(k)-T_m(k)\}\{12(1-n)^2e^{-2kf}-4(1-n)(n+5)+12(n+2)(1-n)kf$$

$$-8(1-n)(2+n)k^2f^2\}+\frac{1}{\mu_1}W_m(k)\{4(1-n)^2e^{-2kf}-(1-n)(5n+1)$$

$$-4(1-n)(2+n)kf+8(1-n)(2+n)k^2f^2\}]\times\Delta^{-1}$$

$$R_{1m}=\frac{S_m(k)+T_m(k)}{\mu_1\left\{1-\frac{1-n}{1+n}e^{-2kf}\right\}},\qquad R'_{1m}=\frac{S_m(k)+T_m(k)}{\mu_1\left\{\frac{1+n}{1-n}e^{2kf}-1\right\}},$$

$$\Delta=-8(n+2)(2n+1)e^{2kf}-16(1-n)^2e^{-2kf}+8(1-n)(5n+4)+64(1-n)kf+32n(1-n)k^2f^2$$

此等決定された常數を (8) に代入し k について零から ∞ 迄積分して變位が得られる．即ち

$$u_r=\int_0^\infty[\{(P_{1m}z-Q_{1m})e^{-kz}+(-P'_{1m}z-Q'_{1m})e^{kz}\}J'_m(kr)$$

$$+(R_{1m}e^{-kz}+R'_{1m}e^{kz})\frac{m}{kr}J_m(kr)]dk.\cos m\theta.$$

$$u_\theta=\int_0^\infty[\{(-P_{1m}z+Q_{1m})e^{-kz}+(P'_{1m}z+Q'_{1m})e^{kz}\}\frac{m}{kr}J_m(kr)$$

$$-(R_{1m}e^{-kz}+R'_{1m}e^{kz})J'_m(kr)]dk.\sin m\theta.$$

$$u_z=\int_0^\infty[-(P_{1m}z-S_{1m})e^{-kz}+(P'_{1m}z-S'_{1m})e^{kz}]J_m(kr)dk.\cos m\theta.$$

今一つの場合は唯 S_m, T_m, W_m が $\overline{S}_m$, $\overline{T}_m$, $\overline{W}_m$ となるだけである．